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Stability of the Damped Mathieu Equation With Time Delay

In the space of the system parameters, the stability charts are determined for the delayed and damped Mathieu equation defined as $\ddot{x}(t)+\kappa\dot{x}(t)+(\delta+\varepsilon\cos t)x(t)=bx(t-2\pi)$*. This stability chart makes the connection between the Strutt-Ince chart of the damped Mathieu equation and the Hsu-Bhatt-Vyshnegradskii chart of the autonomous second order delay-differential equation. The combined charts describe the intriguing stability properties of an important class of delayed oscillatory systems subjected to parametric excitation.* [DOI: 10.1115/1.1567314]

Keywords: Parametric Excitation, Time Delay, Stability

1 Mathematical and Historical Backgrounds

Dynamic problems are often composed in the form of differential equations. The qualitative analysis of these differential equations, and that of the corresponding dynamic phenomena, can be supported by stability charts that show the stability of the system for a range of system parameters.

In this paper, the stability chart of the delayed damped Mathieu equation

$$\ddot{x}(t)+\kappa\dot{x}(t)+(\delta+\varepsilon\cos t)x(t)=bx(t-2\pi) \tag{1}$$

is constructed. This equation combines the effect of parametric excitation on the delayed and damped oscillator.

The three special cases $b=0$, $\varepsilon=0$, and $\kappa=0$ are known from the literature [1–3]. These cases will be overviewed briefly in the following subsections.

1.1 Time Periodic Systems. Parametric excitation often occurs in mechanical systems, when some characteristic properties of the system change periodically in time. The vibrations of rotating shafts with non-symmetric cross-section, the dynamic behavior of gears, or vibrations in belt drives of machine tools are all described by time periodic systems.

The general form of linear periodic ordinary differential equations (ODEs) reads

$$\dot{\mathbf{y}}(t)=\mathbf{A}(t)\mathbf{y}(t), \quad \mathbf{A}(t)=\mathbf{A}(t+T) \tag{2}$$

Here, the coefficient matrix is time periodic.

For periodic ODEs, stability condition is provided by the Floquet Theory [4]. If $\mathbf{y}(T)=\mathbf{\Phi}\mathbf{y}(0)$, then $\mathbf{\Phi}$ is called principle matrix, monodromy matrix or Floquet transition matrix. The eigenvalues of $\mathbf{\Phi}$ are the characteristic multipliers μ_j $(j=1,2,\ldots,n)$ calculated from

$$\det(\mu\mathbf{I}-\mathbf{\Phi})=0 \tag{3}$$

If μ is a characteristic multiplier, and $\mu=\exp(\lambda T)$, then λ is called characteristic exponent [5].

The trivial solution $\mathbf{y}(t)\equiv\mathbf{0}$ of system (2) is asymptotically stable, if and only if all the characteristic multipliers are in modulus less than one, that is, all the characteristic exponents have negative real parts.

Three basic types of stability losses can be classified according to the location of the critical characteristic multipliers.

1. The critical characteristic multipliers are a complex pair moving out of the unit circle, i.e., $|\mu|=1$ and $|\bar{\mu}|=1$ in the critical case. This case is topologically equivalent to the Hopf bifurcation of autonomous nonlinear systems and called *secondary Hopf* or *Neimark-Sacker* bifurcation of a corresponding nonlinear system.
2. The critical characteristic multiplier is real and moves outside the unit circle at $+1$. The arising bifurcation is topologically equivalent to the saddle-node bifurcation of autonomous nonlinear systems and called *period one* bifurcation of a corresponding nonlinear system.
3. The critical characteristic multiplier is real and moves outside the unit circle at -1. There is no topologically equivalent type of bifurcation for autonomous nonlinear systems. This case is called *period two* or *period doubling* or *flip* bifurcation of a corresponding nonlinear system.

Generally, for periodic systems, stability criteria cannot be given in closed form, only approximation methods can be used. Such an approximation method is the Hill's infinite determinant method developed by Hill [6] and Rayleigh [7]. The most straightforward and less accurate method is the piecewise constant approximation of the coefficient matrix [8,9]. There are other methods described in the book of Nayfeh and Mook [10]: the Lindstedt-Poincare technique and the method of multiple scales. A novel approach, the method of Chebyshev polynomials, was developed by Sinha and Wu [11] and improved by Sinha and Butcher [12]. Bauchau and Nikishkov [13] worked out a numerical algorithm for extracting the dominant characteristic multipliers without the explicit computation of the principal matrix. They applied their method for rotorcraft stability evaluation.

Example: The Damped Mathieu Equation. The case $b=0$ of Eq. (11) gives the traditional damped Mathieu equation:

$$\ddot{x}(t)+\kappa\dot{x}(t)+(\delta+\varepsilon\cos t)x(t)=0. \tag{4}$$

This equation was first discussed by Mathieu [14] in connection with the problem of vibrations of an elliptic membrane. Stephenson [15] used an approximate Mathieu equation, and proved, that it is possible to stabilize the upper position of a rigid pendulum by vibrating its pivot point vertically at a specific high frequency.

The stability chart of the Mathieu equation (4), the so called Strutt-Ince diagram was first published by van der Pol and Strutt [1] in 1928. In Fig. 1, the Strutt-Ince diagram is shown for $\kappa=0$, 0.1 and 0.2. For $\kappa<0$, the system is always unstable. The domains denoted by S refer to a stable system, the domains denoted by $U_{\pm1}$ refer to an unstable system. At the stability curves bounding the domains U_{+1} and U_{-1}, there are period one and period two instability, respectively.

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Contributed by the Dynamic Systems and Control Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the ASME JOURNAL OF DYNAMIC SYSTEMS, MEASUREMENT, AND CONTROL. Manuscript received by the ASME Dynamic Systems and Control Division, May 2002; final revision, December 2002. Associate Editor: N. Olgac.

1.2 Delayed Systems. It has been known for a long time, that several problems can be described by models including past effects. One of the classical examples is the predator-prey model of Volterra [16], where the growth rate of predators depends not only on the present quality of food (say, prey), but also on the past quantities (in the period of gestation, say). The first delay models in engineering appeared for wheel shimmy by von Schlippe and Dietrich [17], and for ship stabilization by Minorsky [18].

One of the most important mechanical applications is the cutting process dynamics. After the extensive work of Tlusty et al. [19], Tobias [20] and Kudinov [21,22], the so-called regenerative effect has become the most commonly accepted explanation for machine tool chatter [23,24]. This effect is related to the cutting force variation due to the wavy workpiece surface cut in the previous revolution.

Delayed equations also arise in robotics applications, e.g. telemanipulation with information delay can be mentioned [25–27]. Time delay also arises in neural network models, where the interactions of the neurons are delayed [28].

The systems, where the rate of change of state is determined by the present and also by discrete past states of the system, are described by retarded differential-difference equations (RDDEs). The initial-value problem of general RDDEs was first correctly formulated by Myshkis [29]. Since then, several books appeared summarizing the most important theorems, like the books of Myshkis [30], Bellman and Cooke [31], Halanay [32], Hale [33], Kolmanovskii and Nosov [34], Stépán [23], Hale and Lunel [35], and Diekmann et al. [36].

A linear autonomous RDDE with a single delayed term has the form

$$\dot{\mathbf{y}}(t)=\mathbf{A}\mathbf{y}(t)+\mathbf{B}\mathbf{y}(t-\tau) \tag{5}$$

where $\mathbf{A}$ and $\mathbf{B}$ are $n\times n$ matrices and $\tau>0$. The characteristic function of system (5) reads

$$\det(\lambda\mathbf{I}-\mathbf{A}-\mathbf{B}e^{-\lambda\tau})=0 \tag{6}$$

Opposite to the characteristic polynomial of autonomous ODEs, this characteristic function has, in general, infinite number of zeros. The sufficient and necessary condition for asymptotic stability of (5) is that all the infinite number of characteristic roots have negative real parts.

The first attempts for determining stability criteria for second-order RDDEs was made by Bellman and Cooke [31] and Bhatt and Hsu [37]. They used the D-subdivision method [38] combined with a theorem of Pontryagin [39]. A more sophisticated method was developed by Stépán [23] applicable even for the combination of several discrete and continuous time delays. A novel approach was developed by Olgac and Sipahi [40] for linear systems with a single delay.

Example: The Delayed Oscillator. The case $\varepsilon=0$ of Eq. (11) gives the second order delayed oscillator

$$\ddot{x}(t)+\kappa\dot{x}(t)+\delta x(t)=bx(t-2\pi) \tag{7}$$

Although the stability chart (see Fig. 2) in the parameter plane (δ,b) has a very clear structure, it was first published correctly only in 1966 by Hsu and Bhatt [2]. According to Kolmanovskii and Nosov [34], this chart was also published in the literature in Russian, often referred there as Vyshnegradskii diagram. For the case $\kappa=0$, the stability boundaries are lines with slope $+1$ and -1. For $\kappa=0.1$ and 0.2, the stability boundaries are not lines any more. The $\delta=b$ line is associated to saddle-node instability, all the other boundary curves represent Hopf instabilities.

1.3 Time Periodic Delayed Systems. A linear periodic RDDE with a single delayed term has the form

$$\dot{\mathbf{y}}(t)=\mathbf{A}(t)\mathbf{y}(t)+\mathbf{B}(t)\mathbf{y}(t-\tau),\quad \mathbf{A}(t+T)=\mathbf{A}(t),$$
$$\mathbf{B}(t+T)=\mathbf{B}(t) \tag{8}$$

The Floquet theorem can be extended for these systems as it was shown by Halanay [41], but an infinite dimensional linear operator, the so-called monodromy operator, is defined instead of the finite dimensional fundamental matrix of the traditional Floquet theory [5,33]. This operator can be defined by $\mathbf{y}_T=\mathbf{U}\mathbf{y}_0$, where the continuous function $\mathbf{y}_t$ is defined by the shift $\mathbf{y}_t(\vartheta)=\mathbf{y}(t+\vartheta)$, $\vartheta\in[-\tau,0]$, and T is the principal period of system (8).

The nonzero elements of the spectrum of $\mathbf{U}$ are called the characteristic multipliers of system (8), also defined by

$$\mathrm{Ker}(\mu\mathbf{I}-\mathbf{U})\neq\{\mathbf{0}\} \tag{9}$$

instead of (3). Similarly to the periodic systems, if μ is a characteristic multiplier, and $\mu=\exp(\lambda T)$, then λ is called characteristic exponent.

The trivial solution of system (8) is asymptotically stable, if and only if all the (infinite number of) characteristic multipliers are in modulus less than one, that is all the characteristic exponents have negative real parts. Similarly to time periodic ODEs, the three types of stability losses can be identified according to the location of the critical characteristic multipliers: the *secondary Hopf*, the *period one*, and the *period two* instability routes.

For periodic RDDEs, the operator $\mathbf{U}$ has no closed form, so no closed form stability conditions can be expected. For practical calculations, only approximations can be applied. An alternative of Hill's infinite determinant method was used by Seagalman and Butcher [42] to determine stability properties of turning processes with harmonic impedance modulation. Another approach was used by Insperger and Stépán [43] when the discrete time delay is

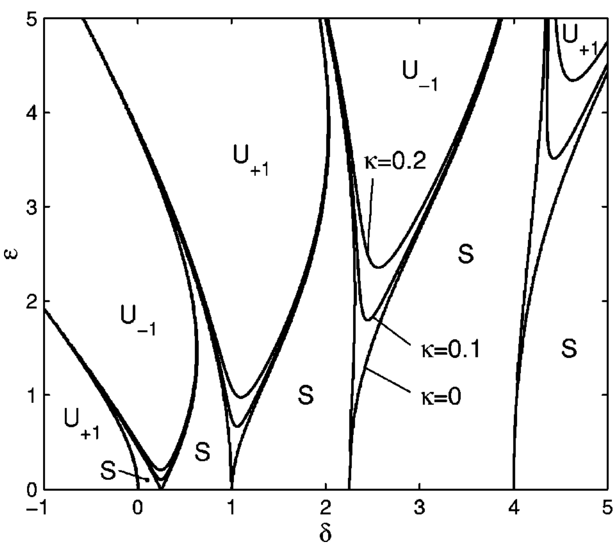

Fig. 1 Strutt-Ince stability chart of the damped Mathieu equation (4)

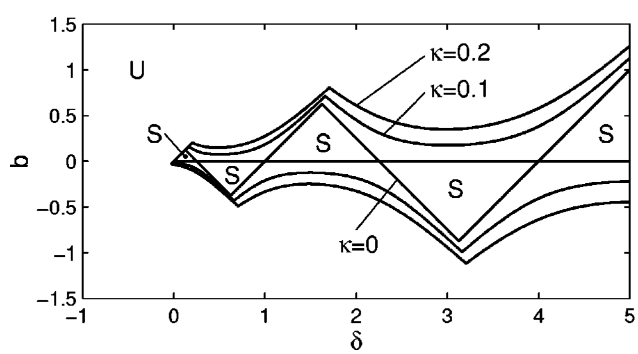

Fig. 2 Hsu-Bhatt-Vyshnegradskii stability chart of Eq. (7)

approximated by special continuous ones, and the infinite dimensional eigenvalue problem is transformed into an approximate finite dimensional one. The time finite element method was developed by Bayly et al. [44] and applied for interrupted cutting processes. Insperger and Stépán [45] developed the so-called semi-discretization method for the approximate stability investigation of general time periodic delayed systems, like equations containing distributed time delay or multiple time delays. Numerical simulation is also a possible way for predicting stability properties [46,47].

Example: The Delayed Mathieu Equation. The case $\kappa=0$ of Eq. (11) gives the delayed Mathieu equation

$$\ddot{x}(t)+(\delta+\varepsilon\cos t)x(t)=bx(t-2\pi) \tag{10}$$

The stability chart of this equation was constructed by Insperger and Stépán [3]. Their work was based on the general theorem, that the number $\mu=e^{\lambda T}$ is a characteristic multiplier of system (8), if and only if, there exists a nontrivial solution in the form $\mathbf{y}(t)=\mathbf{p}(t)e^{\lambda t}$, where $\mathbf{p}(t)=\mathbf{p}(t+T)$. They showed analytically that for any ε, the boundary curves in the plane (δ,b) are straight lines shifted along the boundary curves of the Strutt-Ince diagram. For $\varepsilon=1$, the stability chart in the plane (δ,b) can be seen in Fig. 3, where dashed lines refer to period two loss of stability, continuous lines refer to period one loss of stability. A domain denoted by S refers to an asymptotically stable system, while U refers to instability. The frame-view of the 3-dimensional stability chart in the space (δ,b,ε) is shown in Fig. 4.

2 Delayed Damped Mathieu Equation: Analytical Investigation

The equation of our interest is the delayed damped Mathieu equation

$$\ddot{x}(t)+\kappa\dot{x}(t)+(\delta+\varepsilon\cos t)x(t)=bx(t-2\pi) \tag{11}$$

The special cases $b=0$, $\varepsilon=0$, and $\kappa=0$ was introduced in the previous section. Here, the general case $b\neq 0$, $\varepsilon\neq 0$, and $\kappa\neq 0$ is investigated. Still, Eq. (11) is also special in the sense, that the time delay is just equal to the time period of the parametric excitation. Lots of applications, like milling operations, satisfy this condition.

2.1 Hill's Infinite Determinant Method. Use the trial solution according to the Floquet theorem of RDDEs in the form

$$x(t)=p(t)e^{\lambda t}+\overline{p}(t)e^{\overline{\lambda}t} \tag{12}$$

where $p(t)=p(t+2\pi)$ is a periodic function. Note, that λ is characteristic exponent, that is, if $\mathrm{Re}\,\lambda<0$, then the solution $x(t)\equiv 0$ is asymptotically stable. Expand the periodic function $p(t)$ in (12) into Fourier series

$$x(t)=\left(\sum_{k=0}^{\infty}A_k e^{ikt}+B_k e^{-ikt}\right)e^{\lambda t}+\left(\sum_{k=0}^{\infty}\overline{A}_k e^{-ikt}+\overline{B}_k e^{ikt}\right)e^{\overline{\lambda}t} \tag{13}$$

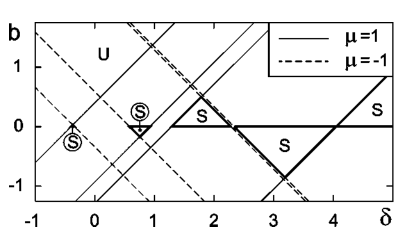

Fig. 3 Domains of stability of Eq. (10) for $\varepsilon=1$

Using trigonometrical transformations, expression (13) can be transformed into the form

$$x(t)=\sum_{k=-\infty}^{\infty}C_k e^{(\lambda+ik)t}+\overline{C}_k e^{(\overline{\lambda}-ik)t} \tag{14}$$

The substitution into the system (11), and the balancing of the harmonics $e^{(\lambda+ik)t}$ and $e^{(\overline{\lambda}-ik)t}$ yield two systems of equations for the coefficients C_k and $\overline{C}_k$, respectively:

$$\frac{\varepsilon}{2}C_{k-1}+c_kC_k+\frac{\varepsilon}{2}C_{k+1}=0,\quad k=-\infty,\ldots,\infty \tag{15a}$$

$$\frac{\varepsilon}{2}\overline{C}_{k-1}+\overline{c_k}\overline{C}_k+\frac{\varepsilon}{2}\overline{C}_{k+1}=0,\quad k=-\infty,\ldots,\infty \tag{15b}$$

where

$$c_k=\delta+(\lambda+ik)^2+\kappa(\lambda+ik)-be^{-2\pi(\lambda+ik)} \tag{16}$$

Equations (15*a*) and (15*b*) are satisfied if and only if λ is a characteristic exponent. Equations (15*a*) and (15*b*) are equivalent, so it is satisfactory to analyze (15*a*) only. There is a nontrivial solution of system (15*a*), if the number zero is an eigenvalue of the so-called Hill's infinite matrix

$$\mathbf{H}(\lambda,\delta,b,\varepsilon)=\begin{pmatrix}\ddots & \ddots & \ddots & \ddots & & & \\ \ddots & \varepsilon/2 & c_{-1} & \varepsilon/2 & 0 & & \\ & 0 & \varepsilon/2 & c_0 & \varepsilon/2 & 0 & \\ & & 0 & \varepsilon/2 & c_1 & \varepsilon/2 & \ddots \\ & & & \ddots & \ddots & \ddots & \ddots\end{pmatrix} \tag{17}$$

This matrix represents an unbounded linear operator $\mathbf{H}:l_2^{\mathbb{Z}}\to l_2^{\mathbb{Z}}$. Here, $l_2^{\mathbb{Z}}$ is the Hilbert space of the complex sequences $(\ldots,z_{-1},z_0,z_1,\ldots)$ with $\sum_{k=-\infty}^{\infty}|z_k|^2<\infty$. As it is the case for (unbounded) linear operators with compact resolvents, the spectrum of $\mathbf{H}$ consists of a countable number of eigenvalues. All of these eigenvalues are of finite multiplicity. The number zero is an eigenvalue of $\mathbf{H}$ if and only if

$$\mathrm{Ker}\ \mathbf{H}(\lambda,\delta,b,\varepsilon)\neq\{\mathbf{0}\} \tag{18}$$

Formula (18) can be treated as the characteristic equation of (11), since its roots are the characteristic exponents. This is a reformulation of (9) with $\mu=\exp(2\pi\lambda)$.

In order to carry out calculations, only the truncated system of equations with $k=-N,\ldots,N$ is considered. This reduces the infinite eigenvalue problem of operator $\mathbf{H}$ to the calculation of a finite determinant

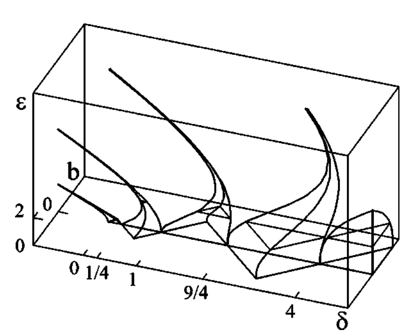

Fig. 4 Stability chart of delayed Mathieu equation (10)

$$D(\lambda,\delta,b,\varepsilon)=\det\begin{pmatrix} c_{-N} & \varepsilon/2 & & & \\ \varepsilon/2 & c_{-N+1} & \varepsilon/2 & & \\ & \ddots & \ddots & \ddots & \\ & & \varepsilon/2 & c_{N-1} & \varepsilon/2 \\ & & & \varepsilon/2 & c_N \end{pmatrix}=0 \tag{19}$$

Although this truncation seems to be a rough approximation, it still has a sound mathematical basis [48,49]. This approximation is just the same as the one applied during the construction of the Strutt-Ince diagram. The operator **H** is often called Hill's infinite matrix, and the terminology *infinite determinant* is also used, although, in fact, it is not a determinant of a matrix.

2.2 Linear Boundary Curves. The system is at the border of stability, if the relevant characteristic exponent is pure imaginary: $\lambda=\mathrm{i}\omega$, where ω is called frequency parameter.

It was shown by Insperger and Stépán [3] that for the case $\kappa=0$, $b\neq 0$, Eq. (19) can be satisfied, if and only if $\omega=j/2$, $j=0,1,\ldots$, and all the boundary curves are straight lines related to period one or period two instabilities.

If $\kappa\neq 0$, then the proof constructed for the undamped case in [3] cannot be used. In this case, Eq. (19) can be satisfied for frequency parameters $\omega\neq j/2$, $j=0,1,\ldots$ as well, and the relevant characteristic multipliers $\mu=\exp(\mathrm{i}2\pi\omega)$ can be complex numbers. Consequently, additional non-straight boundary curves relating to secondary Hopf instabilities may also exist. However, the boundaries related to the frequency parameter $\omega=j/2$, $j=0,1,\ldots$ can be investigated in the same way as it was done in [3].

If j is even, that is $j=2h$, $h=0,1,\ldots$, then $\lambda=\mathrm{i}h$ and the corresponding characteristic multiplier is

$$\mu=\mathrm{e}^{\mathrm{i}h2\pi}=\mathrm{e}^{\mathrm{i}2\pi}=1 \tag{20}$$

In this case, $c_k=\delta-b-(k+h)^2+\mathrm{i}(k+h)\kappa$, and Eq. (19) gives the relation $f_{+1}(\delta-b,\varepsilon,\kappa)=0$ for the boundary curves. For the case $b=0$, the relation $f_{+1}(\delta,\varepsilon,\kappa)=0$ serves the $\mu=+1$ stability boundary curves of the classical damped Mathieu equation defined in the form $\delta=g_{+1}(\varepsilon,\kappa)$. This means, that straight boundary curves exist for the $b\neq 0$ case, in the form $\delta-b=g_{+1}(\varepsilon,\kappa)$. In the plane ($\delta$, b), these are lines with slope $+1$ (see the continuous lines in Fig. 5). Along these boundary curves, there exists a characteristic multiplier $\mu=+1$, and Eq. (11) has a periodic solution of period 2π. This case corresponds to the period one instability route.

If j is odd, that is $j=2h+1$, $h=0,1,\ldots$, then $\lambda=\mathrm{i}(h+1/2)$ and the corresponding characteristic multiplier is

$$\mu=\mathrm{e}^{\mathrm{i}(h+1/2)2\pi}=\mathrm{e}^{\mathrm{i}\pi}=-1 \tag{21}$$

In this case, $c_k=\delta+b-(k+h+1/2)^2+\mathrm{i}(k+h+1/2)\kappa$, and Eq. (19) implies the boundary curve relation $f_{-1}(\delta+b,\varepsilon,\kappa)=0$. For the same reason as above, boundary curves exist again in the form $\delta+b=g_{-1}(\varepsilon,\kappa)$, where $\delta=g_{-1}(\varepsilon,\kappa)$ gives the $\mu=-1$ stability boundary curves of the classical damped Mathieu equation. These boundary curves are straight lines with slope -1 in the parameter plane (δ, b) (see the dashed lines in Fig. 5). Along these boundary curves, there exists a characteristic multiplier $\mu=-1$, and Eq. (11) has nontrivial periodic solution of period 4π. This case corresponds to the period two instability route.

This investigation shows that all the period one and period two boundary curves are straight lines in the (δ, b) plane with slope $+1$ or -1, respectively (see Fig. 5). However, in addition to these linear boundaries, secondary Hopf type boundary curves may also exist related to the cases $\omega\neq j/2$, $j=0,1,\ldots$, as it was explained above. These curves are determined in the following section by the so-called semi-discretization method.

3 Numerical Investigation by Semi-Discretization

In this section, the semi-discretization method [45] is used to construct the stability chart of Eq. (11).

The first step of semi-discretization is the construction of time interval division (t_i, t_{i+1}) of length Δt, $i=0,1,\ldots$ so that $2\pi=(m+1/2)\Delta t$, where m is called approximation parameter. In the ith interval, Eq. (11) can be approximated as

$$\ddot{x}(t)+\kappa\dot{x}(t)+(\delta+\varepsilon c_i)x(t)=bx_{i-m} \tag{22}$$

where

$$c_i=\frac{1}{\Delta t}\int_{t_i}^{t_{i+1}}\cos(t)\mathrm{d}t \tag{23}$$

and

$$x_{i-m}=x(t_{i-m})=x(t_i-m\Delta t) \tag{24}$$

That is, the time periodic coefficient is approximated by a piecewise constant one, and the time delayed term is approximated by a piecewise discrete value. This corresponds to a saw-like approximation of the continuous time delay shown in Fig. 6.

For the initial conditions $x(t_i)=x_i$, $\dot{x}(t_i)=\dot{x}_i$, the solution and its derivative at each time instant t_{i+1} can be determined:

$$x_{i+1}=x(t_{i+1})=a_{00}x_i+a_{01}\dot{x}_i+b_{0m}x_{i-m} \tag{25}$$

$$\dot{x}_{i+1}=\dot{x}(t_{i+1})=a_{10}x_i+a_{11}\dot{x}_i+b_{1m}x_{i-m} \tag{26}$$

where

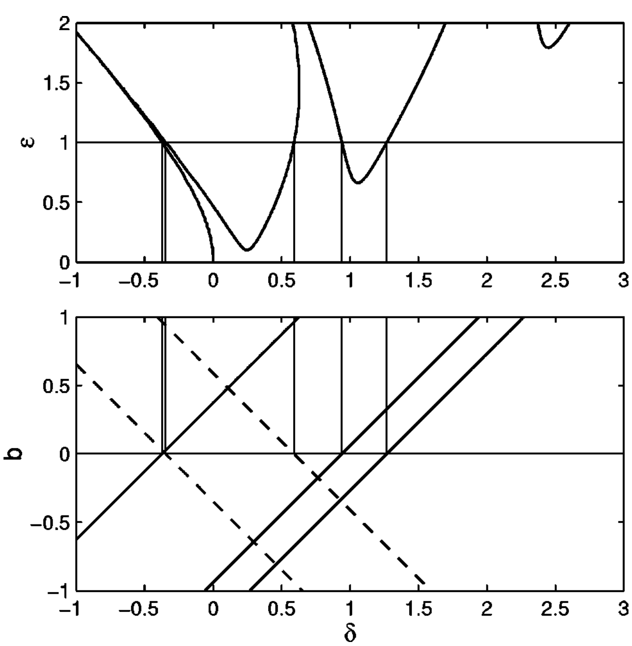

Fig. 5 Period one (continuous) and period two (dashed) boundary lines for Eq. (11) with $\varepsilon=1$, $\kappa=0.1$

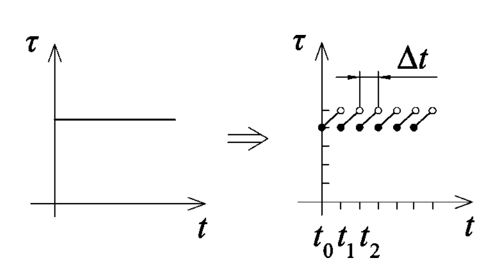

Fig. 6 Approximation of the time delay for $m=4$

$$a_{00}=\kappa_{10}\exp(\lambda_1\Delta t)+\kappa_{20}\exp(\lambda_2\Delta t)$$
$$a_{01}=\kappa_{11}\exp(\lambda_1\Delta t)+\kappa_{21}\exp(\lambda_2\Delta t)$$
$$a_{10}=\kappa_{10}\lambda_1\exp(\lambda_1\Delta t)+\kappa_{20}\lambda_2\exp(\lambda_2\Delta t)$$
$$a_{11}=\kappa_{11}\lambda_1\exp(\lambda_1\Delta t)+\kappa_{21}\lambda_2\exp(\lambda_2\Delta t)$$
$$b_{0m}=\sigma_1\exp(\lambda_1\Delta t)+\sigma_2\exp(\lambda_2\Delta t)+b/(\delta+\varepsilon c_i)$$
$$b_{1m}=\sigma_1\lambda_1\exp(\lambda_1\Delta t)+\sigma_2\lambda_2\exp(\lambda_2\Delta t)$$

and

$$\lambda_{1,2}=\frac{-\kappa\pm\sqrt{\kappa^2-4(\delta+\varepsilon c_i)}}{2},$$

$$\kappa_{10}=\frac{\lambda_2}{\lambda_2-\lambda_1},\quad \kappa_{11}=\frac{-1}{\lambda_2-\lambda_1},\quad \sigma_1=\frac{-\lambda_2}{\lambda_2-\lambda_1}\frac{b}{\delta+\varepsilon c_i},$$

$$\kappa_{20}=\frac{-\lambda_1}{\lambda_2-\lambda_1},\quad \kappa_{21}=\frac{1}{\lambda_2-\lambda_1},\quad \sigma_2=\frac{\lambda_1}{\lambda_2-\lambda_1}\frac{b}{\delta+\varepsilon c_i}$$

Equations (25) and (26) define the discrete map

$$\mathbf{y}_{i+1}=\mathbf{B}_i\mathbf{y}_i, \tag{27}$$

where the $m+2$ dimensional state vector is

$$\mathbf{y}_i=\mathrm{col}(\dot{x}_i \;\; x_i \;\; x_{i-1} \;\; \ldots \;\; x_{i-m}) \tag{28}$$

and the coefficient matrix has the form

$$\mathbf{B}_i=\begin{pmatrix} a_{11} & a_{10} & 0 & \ldots & 0 & b_{1m} \\ a_{01} & a_{00} & 0 & \ldots & 0 & b_{0m} \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \tag{29}$$

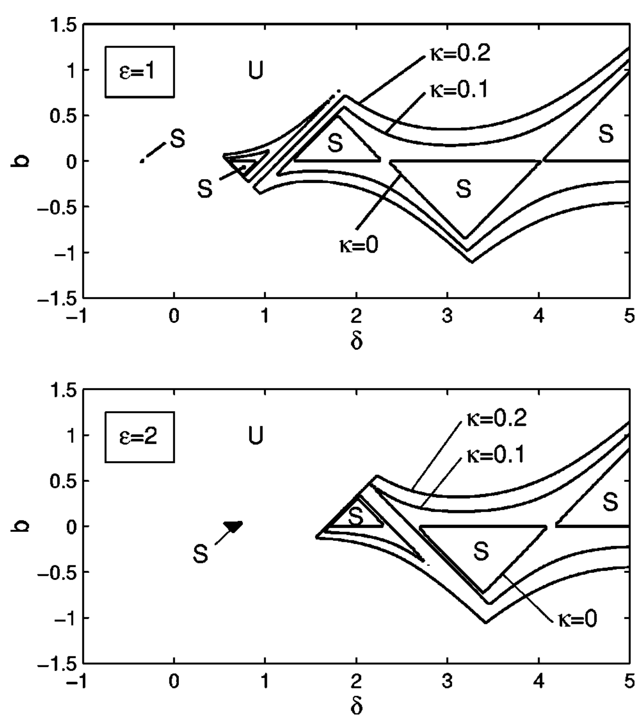

Fig. 7 Stability boundaries for the Eq. (11) obtained by the semi-discretization method

So, the connection between the states at t_i and t_{i+1} is determined by the transition matrix $\mathbf{B}_i$. The transition matrix between the states at t_s and t_f can be given as

$$\mathbf{\Phi}(t_s,t_f)=\mathbf{B}_{f-1}\mathbf{B}_{f-2}\ldots\mathbf{B}_{s+1}\mathbf{B}_s \tag{30}$$

A transition matrix between the states at t_0 and $t_0+2\pi$ would give a finite dimensional approximation of the monodromy operator of Eq. (11). Since $t_0+2\pi=t_0+m\Delta t+\Delta t/2$, the transition matrix $\mathbf{\Phi}(t_0,t_{m+1/2})$ cannot be given in the form of (30), only the approximating transition matrices $\mathbf{\Phi}(t_0,t_m)$ or $\mathbf{\Phi}(t_0,t_{m+1})$ can be used. Note, that these matrices are not principal matrices, since they give the connection between the states at t_0 and $t_0+2\pi-\Delta t/2$ or $t_0+2\pi+\Delta t/2$, and not between t_0 and $t_0+2\pi$. The approximate condition of asymptotic stability is that all the eigenvalues of these matrices are in modulus less than one.

The transition matrix between the states instant t_0 and t_{2m+1} can be given as $\mathbf{\Phi}(t_0,t_{2m+1})=\mathbf{B}_{2m}\mathbf{B}_{2m-1}\ldots\mathbf{B}_1\mathbf{B}_0$. This is a transition matrix over the double principle period, that is $\mathbf{\Phi}(t_0,t_{2m+1})=\mathbf{\Phi}^2(t_0,t_{m+1/2})$. Consequently, the eigenvalues of $\mathbf{\Phi}(t_0,t_{2m+1})$ give the square of the eigenvalues of $\mathbf{\Phi}(t_0,t_{m+1/2})$. Since $|\mu|<1$ if and only if $|\mu^2|<1$, the stability condition for $\mathbf{\Phi}(t_0,t_{2m+1})$ is the same as for the matrices $\mathbf{\Phi}(t_0,t_m)$ or $\mathbf{\Phi}(t_0,t_{m+1})$.

The proof of the convergence of the semi-discretization method is given in [45].

The closed form stability chart [3] of the undamped ($\kappa=0$) case serves as a basis to check the semi-discretization method. A comparison of the stability charts obtained by the eigenvalue investigation of the transition matrices $\mathbf{\Phi}(t_0,t_m)$, $\mathbf{\Phi}(t_0,t_{m+1})$, and $\mathbf{\Phi}(t_0,t_{2m+1})$ shows, that the best convergence is given by the analysis of the matrix $\mathbf{\Phi}(t_0,t_{2m+1})$. The critical eigenvalue of $\mathbf{\Phi}(t_0,t_{2m+1})$ is 1 for both the period-one or period-two cases. So, the two cases can be distinguished only by the analysis of either $\mathbf{\Phi}(t_0,t_m)$ or $\mathbf{\Phi}(t_0,t_{m+1})$.

With a reasonable approximation parameter $m=20$, the infinite dimensional delayed Eq. (11) is approximated by a 22 dimensional discrete system. The eigenvalue analysis of the transition matrix $\mathbf{\Phi}(t_0,t_{2m+1})$ resulted the stability boundaries shown in Fig. 7.

If we compare the exact stability chart in Fig. 3 to the stability chart obtained by the semi-discretization method in Fig. 7 for the undamped reference case $\kappa=0$ and $\varepsilon=1$, the approximation error of the stability boundaries turns out to be less than 1% (within line thickness) for the presented parameter domain with approximation parameter $m=20$. In [45], the convergence of the semi-discretization method was presented for increasing m, that is, the error decreases even further for $m>20$. The same applies for the stability charts of the damped systems with $\kappa>0$. The computation time of one chart in Fig. 7 was in the range of 400 s using MATLAB routines in a 400 MHz PC.

The straight stability boundaries related to period one and period two instabilities show good agreement between the predictions of the Hill's infinite determinant analysis and the results of the semi-discretization method. The charts obtained by the semi-discretization method also confirmed that there exist other non-straight boundary curves related to secondary Hopf instabilities.

4 Conclusions

The delayed damped Mathieu equation was investigated as a basic problem of delayed oscillators subjected to parametric excitation. It was proved, that the delayed damped Mathieu equation also have straight boundary curves with slope $+1$ and -1 in the plane (δ, b) for the period one and period two instabilities, respectively. It was also shown by the semi-discretization method that other non-straight stability boundaries are also inherited from the autonomous system where secondary Hopf loss of stability occur.

Acknowledgments

This research was supported by the Hungarian National Science Foundation under grant no. OTKA TS040792. The authors thank Prof. B. Garay for his helpful discussions on infinite dimensional matrices.

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